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CONFIDENCE INTERVAL PROCEDURES FOR RELIABILITY GROWTH ANALYSIS

LARRY H. CROW

JUNE 1977

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CONFIDENCE INTERVAL PROCEDURES FOR RELIABILITY GROWTH ANALYSIS

By

Larry H. Crow

ABSTRACT

The Weibull Process (a nonhomogeneous Poisson process with intensity $r(t) = \lambda\beta t^{\beta-1}$) is considered as a stochastic model for the Duane [3] reliability growth postulate. Under this model the mean time between failure (MTBF) for the system at time t is given by $M(t) = [r(t)]^{-1}$. Small sample and asymptotic confidence intervals on M(t) are determined for failure and time truncated testing. Tabled values to compute the confidence intervals and numerical examples illustrating these procedures are also presented.

1. INTRODUCTION

In the development of complex systems, the first prototypes produced will invariably contain design and engineering deficiencies. To meet various system performance requirements, such as for reliability, it is commonplace to subject these early prototypes to a "test-fix-retest" process. During this process prototypes are tested to identify deficiencies. Corrections for these deficiencies are incorporated into the prototypes which are then subjected to further testing to verify the fixes and surface new problem areas.

A number of reliability growth models have been proposed in the literature for estimating system reliability during development testing. A model of wide interest in government and industry is the Duane postulate that the instantaneous system MTBF at cumulative test time t be expressed

as $M(t) = [\lambda\beta t^{\beta-1}]^{-1}$, where $0 < \lambda$ and $0 < \beta$ are parameters. (In Duane [3], $K = \lambda$ and $\alpha = 1-\beta$.) This postulate was based on Duane observing that the cumulative failure rate versus cumulative test time fell close to a straight line when plotted on ln-ln scale for five divergent types of systems developed at General Electric.

Crow [2] noted that the Duane postulate could be stochastically represented as a Weibull process, thus allowing for statistical procedures based on this process to be used in the application of the postulate to reliability growth analyses. In particular, a goodness of fit test, maximum likelihood (ML) estimates of λ and β, and confidence interval procedures for the parameters β and $\mu = \lambda^{-1/\beta}$ are presented in [1], [2], [4].

In the application of the Weibull process model to reliability growth, estimates and confidence intervals for the MTBF function $M(t)$ are of considerable practical interest since $M(t)$ represents the reliability status of the system at time t. Estimates of $M(t)$ can be determined directly from estimates of λ and β. In the present paper, we consider small sample and asymptotic confidence intervals on $M(t)$ when data from a Weibull process are failure or time truncated at t. Appropriate tables to compute these confidence intervals are also presented.

2. THE MODEL

Let $0 < X_1 < X_2 < \ldots$ denote the successive failure times, on a cumulative time scale, for a system undergoing development testing and let $N(t)$ denote the number of failures during $(0,t]$. In the formulation of the reliability growth postulate, Duane observed, for the systems under

study, that plots of $N(t)/t$ versus t were approximately linear on ln-ln scale. A probabilistic model for reliability growth reflecting this property exactly for $E(N(t))$ is the nonhomogeneous Poisson process $\{N(t),\ t > 0\}$ with mean value function $E(N(t)) = \lambda t^{\beta}$ and intensity function $r(t) = \frac{dE(N(t))}{dt} = \lambda\beta t^{\beta-1}$. When $\beta = 1$, $r(t) \equiv \lambda$ and the times between successive failures $X_{i+1} - X_i$ follow an exponential distribution with mean $1/\lambda$, indicating no reliability growth. In the presence of reliability growth, however, the times $X_{i+1} - X_i$ should be stochastically increasing. For the Weibull process this occurs when $0 < \beta < 1$, i.e. when $r(t)$ is decreasing (see Parzen [7], Chapter 4).

Suppose modifications are introduced into the system during the time period $(0,t_o]$. At time t_o the intensity of failure is $r(t_o) = \lambda\beta t_o^{\beta-1}$. In practice it is generally assumed that if no improvements are incorporated into the system after time t_o, then failures would continue at the constant rate $r(t_o) = \lambda\beta t_o^{\beta-1}$ with further testing. That is, if no additional modifications are made on the system after time t_o, then the intervals between successive failures $X_{i+1}-X_i$ would follow an exponential distribution with mean $M(t_o) = [\lambda\beta t_o^{\beta-1}]^{-1}$ for $X_i \geq t_o$.

From the above, the function $M(t) = [\lambda\beta t^{\beta-1}]^{-1}$ is interpreted as the instantaneous MTBF of the system at time t and may, therefore, represent the system reliability growth under this model. When t corresponds to the total time the system has been on test, then $M(t)$ is the <u>achieved</u> MTBF or the MTBF of the system in its present configuration.

3. CONFIDENCE INTERVALS FOR ACHIEVED MTBF

In this section we shall present confidence interval procedures for the achieved MTBF for the cases when data are failure and time truncated.

A. Data Failure Truncated

Suppose data from a Weibull process are truncated at the n-th failure yielding observed failure times $X_1 < X_2 < \ldots < X_n$. The ML estimates of λ and β determined from these data are

$$\hat{\lambda} = n/X_n^{\hat{\beta}} \quad \text{and} \quad \hat{\beta} = n/\sum_{i=1}^{n-1} \ln(X_n/X_i). \tag{3.1}$$

Under the reliability growth model considered in this paper, $M(X_n)$ is the achieved MTBF of the system and may be estimated by

$$\hat{M}(X_n) = [\hat{\lambda}\hat{\beta}X_n^{\hat{\beta}-1}]^{-1} = X_n/n\hat{\beta}. \tag{3.2}$$

In the Appendix we show that

$$\frac{n^2\hat{M}(X_n)}{M(X_n)} = Y_{n-1} \cdot Y_n, \tag{3.3}$$

where Y_{n-1} and Y_n are independent random variables and Y_r has the gamma probability density function (p.d.f.)

$$g_r(y) = \frac{y^{r-1}\, e^{-y}}{(r-1)!}, \; y > 0, \; r = 1,2,\ldots \tag{3.4}$$

Using (3.3) and (3.4) we determined by numerical integration values u_p such that $\text{Prob}(M(X_n)/\hat{M}(X_n) < u_p) = p$. These are given in Table 1 for various values of p and n = 2(1)30(5)50(10)80(20)100. The entries in Table 1 are accurate to the four significant numbers given.

Exact $(1-\alpha)100$ percent two-sided confidence intervals on $M(X_n)$ are of the form

$$(\rho_1 \hat{M}(X_n),\ \rho_2 \hat{M}(X_n)) \tag{3.5}$$

where ρ_1 and ρ_2 are from Table 1 such that $\text{Prob}(\rho_1 < M(X_n)/\hat{M}(X_n) < \rho_2) = 1-\alpha$.

From (3.3) and (3.4) it is straightforward to show also that

$$\sqrt{n}\left(\frac{\hat{M}(X_n)}{M(X_n)} - 1\right)$$

is asymptotically normally distributed with mean 0 and standard deviation $\sqrt{2}$ as $n \to \infty$. Thus, for large n approximate $(1-\alpha)100$ percent two-sided confidence intervals are of the form (3.5) where

$$\rho_1 \doteq [\,1+ \sqrt{2/n}\, Z_{\alpha/2}]^{-1}\ ,\ \rho_2 \doteq [\,1- \sqrt{2/n}\, Z_{\alpha/2}]^{-1} \tag{3.6}$$

and $Z_{\alpha/2}$ is the $(1-\alpha/2)$-th percentile for the standard normal distribution.

B. Data Time Truncated

Let T be predetermined and suppose $n \geq 1$ failures are observed for the Weibull process during $(0,T]$ at times $0 < X_1 < X_2 < \ldots < X_n$. The ML estimates of λ and β from these data are

$$\hat{\lambda} = n/T^{\hat{\beta}} \text{ and } \hat{\beta} = n/\sum_{i=1}^{n} \ln(T/X_i) \tag{3.7}$$

and the ML estimate of M(T), the achieved MTBF at time T, is

$$\hat{M}(T) = [\hat{\lambda}\hat{\beta}T^{\hat{\beta}-1}]^{-1} = T/n\hat{\beta}. \qquad (3.8)$$

Let $N(T) = N$ and $W = \sum_{i=1}^{N} \ln(T/X_i)$. (We adopt the usual convention that sums of the form $\sum_{i=1}^{0}$ are equal to zero.) In the Appendix we show that having observed $N=n$, $W=w>0$, a lower $(1-\alpha)100$ percent confidence bound for the parameter $\phi = T/M(T) = \lambda\beta T^{\beta}$ is the value ϕ_1 satisfying

$$\sum_{j=n}^{\infty} \frac{(w\phi_1)^{j-1/2}}{j!(j-1)!I_1(2\sqrt{w\phi_1})} = \alpha \qquad (3.9)$$

where $I(\cdot)$ is the modified Bessel function of order 1. Similarly, it is shown that an upper $(1-\alpha)100$ percent confidence bound on ϕ is the value ϕ_2 satisfying

$$\sum_{j=1}^{n} \frac{(w\phi_2)^{j-1/2}}{j!(j-1)!I_1(2\sqrt{w\phi_2})} = \alpha. \qquad (3.10)$$

A two-sided $(1-\alpha)100$ percent confidence interval for ϕ is determined by setting the right-hand sides of (3.9) and (3.10) equal to $\alpha/2$.

Because of the discreteness of the random variable N, the above bounds are not exact, but are conservative in the sense that the probability of containing ϕ is at least $1-\alpha$.

From confidence bounds ϕ_1 and ϕ_2 on ϕ, the corresponding confidence bounds on M(T) are T/ϕ_2 and T/ϕ_1. To produce a table for calculating

these bounds we found the solutions γ_1 and γ_2 to the equations

$$\sum_{j=n}^{\infty} \frac{\gamma_1^{j-1/2}}{j!(j-1)!I_1(2\sqrt{\gamma_1})} = \alpha \tag{3.11}$$

and

$$\sum_{j=1}^{n} \frac{\gamma_2^{j-1/2}}{j!(j-1)!I_1(2\sqrt{\gamma_2})} = \alpha \tag{3.12}$$

for various values of α and n. From (3.9)-(3.12) we have that $\phi_1 = \gamma_1/w$ and $\phi_2 = \gamma_2/w$. Since $w = n/\hat{\beta}$ it follows that $T/\phi_1 = (n^2/\gamma_1)\hat{M}(T)$ and $T/\phi_2 = (n^2/\gamma_2)\hat{M}(T)$.

In Table 2 we list $\Pi_1 = n^2/\gamma_2$ and $\Pi_2 = n^2/\gamma_1$ for the designated confidence coefficients and n = 2(1)30(5)50(10)80(20)100. These entries are accurate to the three decimal places given. For appropriate Π_1 and Π_2 from Table 2, two-sided $(1-\alpha)100$ percent confidence intervals on M(T) are

$$(\Pi_1\hat{M}(T),\ \Pi_2\hat{M}(T)). \tag{3.13}$$

For approximate confidence intervals for M(T) when the observed number of failures is large we use a result of Harris and Soms [5] to show that conditioned on W=w, $(N-\psi)/\sqrt{\psi}$ is asymptotically normal with mean 0 and standard deviation $1/\sqrt{2}$ as $T \to \infty$, where $\psi = \sqrt{w\lambda\beta T^{\beta}}$. Hence,

for $N=n$ large, approximate two-sided $(1-\alpha)100$ percent confidence intervals on ψ^2 can be determined from

$$\delta_1 = \left[n + \tfrac{1}{2}C^2_{\alpha/2} + \sqrt{nC^2_{\alpha/2} + \tfrac{1}{4}C^4_{\alpha/2}}\right]^2$$

$$\delta_2 = \left[n + \tfrac{1}{2}C^2_{\alpha/2} - \sqrt{nC^2_{\alpha/2} + \tfrac{1}{4}C^4_{\alpha/2}}\right]^2 \tag{3.14}$$

where $C_{\alpha/2} = Z_{\alpha/2}/\sqrt{2}$.

Since $w = n/\hat{\beta}$, then $\psi^2 = n^2\hat{M}(T)/M(T)$. Hence, for large n, approximate two-sided $(1-\alpha)100$ percent confidence intervals on M(T) are of the form (3.13) where

$$\Pi_1 \doteq n^2/\delta_1 \;, \; \Pi_2 \doteq n^2/\delta_2. \tag{3.15}$$

From (3.14) we may further approximate Π_1 and Π_2 by

$$\Pi_1 \doteq n^2/\tau_1 \;, \; \Pi_2 \doteq n^2/\tau_2 \tag{3.16}$$

for large n where

$$\tau_1 = (n + C_{\alpha/2}\sqrt{n})^2 \;, \; \tau_2 = (n - C_{\alpha/2}\sqrt{n})^2.$$

4. EXAMPLES

We shall now illustrate by numerical examples the confidence interval procedures presented in the previous section.

Suppose a system subjected to development testing until the 15-th failure recorded the following successive failure times: .7, 2.4, 8.2, 11.3, 12.1, 17.6, 18.9, 20.4, 21.9, 23.2, 25.7, 42.8, 48.0, 56.3, 65.1. Since the data are failure truncated we use (3.1) to calculate $\hat{\lambda} = .756$ and $\hat{\beta} = .715$. From (3.2) the ML estimate of the achieved system MTBF M(65.1) at time 65.1 is $\hat{M}(65.1) = 6.1$. For a two-sided 90

percent confidence interval on the achieved MTBF we choose from Table 1 for $n = 15$ the values $\rho_1 = .6299$ and $\rho_2 = 2.182$. From (3.5) the two-sided confidence interval is (3.8, 13.3).

We now consider the situation where development testing data for a system were truncated at the predetermined time $T = 500$ hours. During this period the system experienced the following $n = 23$ successive failure times; .2, 4.2, 4.5, 5.0, 5.4, 6.1, 7.9, 14.8, 19.2, 48.6, 85.8, 108.9, 127.2, 129.8, 150.1, 159.7, 227.4, 244.7, 262.7, 315.3, 329.6, 404.3, 486.2. From (3.7) and (3.8) we obtain $\hat{\lambda} = 1.769$, $\hat{\beta} = .413$ and $\hat{M}(500) = 52.7$. To place a two-sided 95 percent confidence interval on the achieved MTBF M(500) we locate the appropriate values in Table 2 for $n = 23$. These are $\Pi_1 = .563$, $\Pi_2 = 1.961$. By (3.13) the confidence interval is (29.7, 103.3)

Let $n = 100$. If data are failure truncated approximate 95 percent two-sided confidence intervals may be determined using (3.6). This gives $\rho_1 \doteq .783$, $\rho_2 \doteq 1.383$. The exact values from Table 1 are $\rho_1 = .776$, $\rho_2 = 1.355$. If data are time truncated, approximate 95 percent two-sided confidence intervals may be determined using the large sample relationships for Π_1 and Π_2 given by (3.15) and (3.16). Using (3.16) we obtain $\Pi_1 \doteq .771$, $\Pi_2 \doteq 1.348$. These compare with the exact values from Table 2 of $\Pi_1 = .758$, $\Pi_2 = 1.347$.

5. APPENDIX

For a nonhomogeneous Poisson process $\{N(t), t>0\}$ with intensity $r(t)$, the probability of an event occurring in an infinitesimally small interval $(t, t + \Delta t]$ is approximately $r(t)\Delta t$. Also, if

$N(s,t) = N(t)-N(s)$, $t>s$, then $N(s,t)$ has a Poisson distribution with mean $E[N(t)]-E[N(s)]$. From these basic properties and the fact that the process has independent increments it follows that the p.d.f. for the first n successive events $X_1 < X_2 < \ldots < X_n$ is

$$h(x_1, x_2,\ldots,x_n) = \prod_{i=1}^{n} r(x_i) \prod_{i=1}^{n} \text{Prob}(N(x_j, x_{j-1}) = 0),$$

where $x_o = 0$. For the Weibull process,

$$h(x_1, x_2,\ldots,x_n, \lambda,\beta) = \lambda^n\beta^n e^{-\lambda x_n} \prod_{i=1}^{n} x_i^{\beta-1}$$

which yields the ML estimates $\hat{\lambda}$, $\hat{\beta}$ given in (3.1).

We also note that the p.d.f. of X_n is given by

$$\begin{aligned} g_n(x,\lambda,\beta) &= \text{Prob}(N(x) = n-1)r(x) \\ &= \frac{(\lambda x^\beta)^{n-1}\, e^{-\lambda x}}{(n-1)!} \lambda\beta x^{\beta-1} \end{aligned} \tag{5.1}$$

and hence, the conditional p.d.f. of $X_1, X_2,\ldots,X_{n-1}$, given $X_n = x_n$ is

$$h(x_1, x_2, \ldots,x_{n-1}, \beta|X_n=x_n) = (n-1)! \prod_{i=1}^{n-1} \beta \frac{x_i^{\beta-1}}{x_n^\beta}.$$

That is, conditioned on $X_n = x_n$, $X_1, X_2,\ldots,X_{n-1}$, unordered, are distributed as n-1 independent random variables with common p.d.f. $\beta x^{\beta-1}/x_n^\beta$. This implies that $n\beta/\hat{\beta}$ is a gamma random variable with p.d.f.

$$g_{n-1}(x) = \frac{x^{n-2}e^{-x}}{(n-2)!}$$

independent of X_n. Also, from (5.1), λX_n^β has the gamma p.d.f.

$$g_n(x) = \frac{x^{n-1}e^{-x}}{(n-1)!}.$$

From the last two results, (3.3) and the asymptotically normality of

$$\sqrt{n}\left(\frac{\hat{M}(X_n)}{M(X_n)} - 1\right)$$

are established.

Let T be fixed and N(T) = N. For time truncated testing the p.d.f. of $X_1, X_2, \ldots, X_N$ is

$$f(x_1, x_2, \ldots, x_n) = \text{Prob}(N(T)-N(x_n) = 0) \prod_{i=1}^{n} r(x_i)\text{Prob}(N(x_i)-N(x_{i-1}) = 0)$$

which, for the Weibull process yields

$$f(x_1, x_2, \ldots, x_n, \lambda, \beta) = \lambda^n \beta^n e^{-\lambda T^\beta} \prod_{i=1}^{n} x_i^{\beta-1}. \tag{5.2}$$

From (5.2) we obtain the likelihood function and the ML estimates $\hat{\lambda}$ and $\hat{\beta}$ given in (3.7).

Observe that (5.2) implies that (N,W) are sufficient statistics for (λ,β) where $W = \sum_{i=1}^{N} \ln(T/X_i)$.

To place confidence bounds on $M(T) = T/\beta\theta$, where $\theta = \lambda T^\beta$, we therefore restrict attention to (N, W). Observe that no information is contained in the sample for β when W=0, or equivalently when N=0. We shall next determine the joint p.d.f. of (N, W) given W>0.

Since N has the Poisson distribution with mean θ it follows from (5.2) that the conditional p.d.f. of $(X_1, X_2, \ldots, X_N)$ given N=n is

$$f(x_1, x_2, \ldots, x_n, \beta | N=n) = n! \prod_{i=1}^{n} \beta x_i^{\beta-1}/T^\beta.$$

That is, the ordered times $X_1, X_2, \ldots, X_N$, conditioned on N=n, are distributed as order statistics for a sample of size n from a distribution with p.d.f. $\beta x^{\beta-1}/T^{\beta}$. Hence, the p.d.f. of W, given $N = n \geq 1$, is

$$q(w,\beta|N=n) = \frac{(\beta w)^{n-1} e^{-\beta w}\ \beta}{(n-1)!}, \quad w>0,\ n=1,2,\ldots$$

Therefore, the p.d.f. of (N, W), given W>0, is

$$p(n,w,\phi,\beta) = \frac{\phi^n w^{n-1} e^{-\beta w-\phi/\beta}}{n!,(n-1)!(1-e^{-\phi/\beta})} \tag{5.3}$$

$w>0$, n = 1, 2,..., where $\phi = T/M(T)$.

To place confidence bounds on ϕ, and hence on M(T), we observe that $p(n,w,\phi,\beta)$ is a member of the exponential family. Thus, we may determine confidence bounds on ϕ by considering the distribution of N given $W = w>0$ (see 6 pp. 134-40). From (5.3) the p.d.f. of N given $W = w>0$ is

$$p(n,\phi|W=w) = \frac{\dfrac{\phi^n w^{n-1} e^{-\beta w-\phi/\beta}}{n!(n-1)!(1-e^{-\phi/\beta})}}{\displaystyle\sum_{j=1}^{\infty} \frac{\phi^j w^{j-1} e^{-\beta w-\phi/\beta}}{j!(j-1)!(1-e^{-\phi/\beta})}} = \frac{(w\phi)^{n-1/2}}{n!(n-1)!I_1(2\sqrt{w\phi})}, \tag{5.4}$$

where $I_1(\cdot)$ is the modified Bessel function of order 1.

Thus, after observing N=n, lower and upper $(1-\alpha)100$ percent confidence bounds on ϕ are the values ϕ_1 and ϕ_2 satisfying (3.9) and (3.10), respectively. Randomized confidence bounds determined by considering the distribution of the random variable M given W= w>0 where M = N + U and U is uniform on (0,1) and independent of N, would have confidence coefficient of exactly $(1-\alpha)$ and would also be uniformly most accurate unbiased confidence intervals.

We shall next show that conditioned on $W = w>0$, $(N-\psi)/\sqrt{\psi}$ is asymptotically normal with mean 0 and standard deviation $1/\sqrt{2}$ as $T \to \infty$, where ψ is defined in Section 3. Let $K = N-1$ and note that from (5.4)

$$\mathrm{Prob}(K \leq k | W=w) = \sum_{j=0}^{k} \frac{(w\phi)^{j+1/2}}{j!(j+1)!I_1(2\sqrt{w\phi})}$$

which is an incomplete modified Bessel distribution. Harris and Soms [5] considered this class of distributions and showed that $(K-\psi)/\sqrt{\psi}$ is asymptotically normal with mean 0 and standard deviation $1/\sqrt{2}$ as $\psi \to \infty$. The desired result follows.

6. ACKNOWLEDGEMENT

The author wishes to thank David L. Clark for the computer programming which generated Tables 1 and 2.

Table 1. Percentage Points U_p such that $\mathrm{Prob}(M(X_n)/\hat{M}(X_n) < U_p) = P$

n \ P	.005	.010	.025	.050	.100	.900	.950	.975	.990	.995
2	.2378	.2944	.4099	.5552	.8065	33.76	72.67	151.5	389.9	788.6
3	.2627	.3119	.4054	.5137	.6840	8.927	14.24	21.96	37.60	55.52
4	.2902	.3368	.4225	.5174	.6601	5.328	7.651	10.65	15.96	21.31
5	.3151	.3603	.4415	.5290	.6568	4.000	5.424	7.147	9.995	12.68
6	.3372	.3815	4595	.5421	.6600	3.321	4.339	5.521	7.388	9.076
7	.3569	4003	.4760	.5548	.6656	2.910	3.702	4.595	5.963	7.162
8	.3746	.4173	.4910	.5668	.6720	2.634	3.284	4.002	5.074	5.993
9	.3906	.4327	.5046	.5780	.6787	2.436	2.989	3.589	4.469	5.211
10	.4052	.4467	.5171	.5883	.6852	2.287	2.770	3.286	4.032	4.652
11	4185	4595	.5285	.5979	.6915	2.170	2.600	3.054	3.702	4.233
12	4308	.4712	.5391	.6067	6975	2.076	2.464	2.870	3.443	3.909
13	.4422	4821	.5488	.6150	.7033	1.998	2.353	2.721	3.235	3.650
14	4528	4923	.5579	.6227	.7087	1.933	2.260	2.597	3.064	3.438
15	4627	.5017	.5664	.6299	.7139	1.877	2.182	2.493	2.921	3.262
16	.4719	.5106	.5743	.6367	.7188	1.829	2.114	2.404	2.800	3.113
17	.4807	.5189	.5818	.6431	.7234	1.788	2.056	2.327	2.695	2.985
18	.4889	.5267	.5888	.6491	.7278	1.751	2.004	2.259	2.604	2.874
19	.4967	.5341	.5954	.6547	.7320	1.718	1.959	2.200	2.524	2.777
20	.5040	.5411	.6016	.6601	.7360	1.688	1.918	2.147	2.453	2.691
21	.5110	.5478	.6076	.6652	.7398	1.662	1.881	2.099	2.390	2.614
22	.5177	.5541	.6132	.6701	.7434	1.638	1.848	2.056	2.333	2.546
23	.5240	.5601	.6186	.6747	.7469	1.616	1.818	2.017	2.281	2.484
24	.5301	.5659	.6237	.6791	.7502	1.596	1.790	1.982	2.235	2.428
25	.5359	.5714	.6286	.6833	.7534	1.578	1.765	1.949	2.192	2.377
26	.5415	.5766	.6333	.6873	.7565	1.561	1.742	1.919	2.153	2.330
27	.5468	.5817	.6378	.6912	.7594	1.545	1.720	1.892	2.116	2.287
28	.5519	.5865	.6421	.6949	.7622	1.530	1.700	1.866	2.083	2.247
29	.5568	.5912	.6462	.6985	.7649	1.516	1.682	1.842	2.052	2.211
30	.5616	.5957	.6502	.7019	.7676	1.504	1.664	1.820	2.023	2.176
35	.5829	.6158	.6681	.7173	.7794	1.450	1.592	1.729	1.905	2.036
40	.6010	.6328	.6832	.7303	.7894	1.410	1.538	1.660	1.816	1.932
45	.6168	.6476	.6962	.7415	.7981	1.378	1.495	1.606	1.747	1.852
50	.6305	.6605	.7076	.7513	.8057	1.352	1.460	1.562	1.692	1.787
60	.6538	.6823	.7267	.7678	.8184	1.312	1.407	1.496	1.607	1.689
70	.6728	.7000	.7423	.7811	.8288	1.282	1.367	1.447	1.546	1.618
80	.6887	.7148	.7553	.7922	.8375	1.259	1.337	1.409	1.499	1.564
100	.7142	.7384	.7759	.8100	.8514	1.225	1.293	1.355	1.431	1.486

Table 2. Values π_1 and π_2 such that $(\pi_1 \hat{M}(T),\ \pi_2 \hat{M}(T))$ are Confidence Intervals for $M(T)$.

n	CONFIDENCE COEFFICIENT							
	.80		.90		.95		.98	
	π_1	π_2	π_1	π_2	π_1	π_2	π_1	π_2
2	.261	18.655	.200	38.661	.159	78.664	.124	198.666
3	.333	6.326	.263	9.736	.217	14.552	.174	24.103
4	.385	4.243	.312	5.947	.262	8.093	.215	11.811
5	.426	3.386	.352	4.517	.300	5.862	.250	8.043
6	.459	2.915	.385	3.764	.331	4.738	.280	6.254
7	.487	2.616	.412	3.298	.358	4.061	.305	5.216
8	.511	2.407	.436	2.981	.382	3.609	.328	4.539
9	.531	2.254	.457	2.750	.403	3.285	.349	4.064
10	.549	2.136	.476	2.575	.421	3.042	.367	3.712
11	.565	2.041	.492	2.436	.438	2.852	.384	3.441
12	.579	1.965	.507	2.324	.453	2.699	.399	3.226
13	.592	1.901	.521	2.232	.467	2.574	.413	3.050
14	.604	1.846	.533	2.153	.480	2.469	.426	2.904
15	.614	1.800	.545	2.087	.492	2.379	.438	2.781
16	.624	1.759	.556	2.029	.503	2.302	.449	2.675
17	.633	1.723	.565	1.978	.513	2.235	.460	2.584
18	.642	1.692	.575	1.933	.523	2.176	.470	2.503
19	.650	1.663	.583	1.893	.532	2.123	.479	2.432
20	.657	1.638	.591	1.858	.540	2.076	.488	2.369
21	.664	1.615	.599	1.825	.548	2.034	.496	2.313
22	.670	1.594	.606	1.796	.556	1.996	.504	2.261
23	.676	1.574	.613	1.769	.563	1.961	.511	2.215
24	.682	1.557	.619	1.745	.570	1.929	.518	2.173
25	.687	1.540	.625	1.722	.576	1.900	.525	2.134
26	.692	1.525	.631	1.701	.582	1.873	.531	2.098
27	.697	1.511	.636	1.682	.588	1.848	.537	2.068
28	.702	1.498	.641	1.664	.594	1.825	.543	2.035
29	.706	1.486	.646	1.647	.599	1.803	.549	2.006
30	.711	1.475	.651	1.631	.604	1.783	.554	1.980
35	.729	1.427	.672	1.565	.627	1.699	.579	1.870
40	.745	1.390	.690	1.515	.646	1.635	.599	1.788
45	.758	1.361	.705	1.476	.662	1.585	.617	1.723
50	.769	1.337	.718	1.443	.676	1.544	.632	1.671
60	.787	1.300	.739	1.393	.700	1.481	.657	1.591
70	.801	1.272	.756	1.356	.718	1.435	.678	1.533
80	.813	1.251	.769	1.328	.734	1.399	.695	1.488
100	.831	1.219	.791	1.286	.758	1.347	.722	1.423

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